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Annealed survival asymptotics for Brownian motion in a scaled Poissonian potential

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Abstract

We consider d-dimensional Brownian motion evolving in a scaled Poissonian potential $\beta\varphi^{-2}(t)V$, where $\beta>0$ is a constant, φ is the scaling function which typically tends to infinity, and V is obtained by translating a fixed non-negative compactly supported shape function to all the particles of a d-dimensional Poissonian point process. We are interested in the large t behavior of the annealed partition sum of Brownian motion up to time t under the influence of the natural Feynman–Kac weight associated to $\beta\varphi^{-2}(t)V$. We prove that for $d\geqslant 2$ there is a critical scale φ and a critical constant $\beta_c(d)>0$ such that the annealed partition sum undergoes a phase transition if β crosses $\beta_c(d)$. In $d=1$ this picture does not hold true, which can formally be interpreted that on the critical scale φ we have $\beta_c(1)=0$.

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0. Introduction and results

0.1. The model

In the present article we study the behavior of d-dimensional Brownian motion under the influence of a scaled random soft potential, $d\geqslant 1$. The random soft potential is obtained by translating a fixed shape function W to all the points of a Poissonian cloud. Let $\mathbb{P}$ stand for the law of a Poissonian point process $\omega=\sum_i \delta_{x_i}\in\Omega$ with fixed intensity $v=1$ (Ω is the set of all simple pure locally finite point measures on $\mathbb{R}^d$). For $\omega\in\Omega$, $x\in\mathbb{R}^d$, the (unscaled) soft Poissonian potential is then defined as

$$V(x,\omega)\stackrel{\text{def}}{=}\int W(x-y)\,\omega(\mathrm{d}y), \tag{0.1}$$

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where we assume that the shape function $W \geqslant 0$ is measurable, bounded, compactly supported, and $\int W(y)\,\mathrm{d}y = 1$. For $x \in \mathbb{R}^d$, let P_x stand for the standard Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting from x (its canonical process is denoted by Z). Then it is well known that the Feynman–Kac functional $u(x,t,\omega) = E_x[\exp\{-\int_0^t V(Z_s,\omega)\,\mathrm{d}s\}]$ represents the bounded solution in a classical sense when $V(\cdot,\omega)$ is regular and in a generalized sense else of the random potential parabolic equation:

$$\partial_t u = \frac{1}{2}\Delta u - V(\cdot,\omega)u,$$
$$u_{t=0} = 1. \tag{0.2}$$

We know that the annealed large t behavior of that solution is

$$\mathbb{E}[u(0,t,\omega)] = \exp\{-\tilde{c}(d,1)t^{d/(d+2)}(1+\mathrm{o}(1))\} \quad \text{as } t \to \infty, \tag{0.3}$$

where $\tilde{c}(d,1)$ is the constant defined in (0.10), below. This result goes back to Donsker and Varadhan (1975), who used large deviation theory for occupation local times of Brownian motion on a torus. In a later version, Sznitman (1998, Theorem 4.5.3), has proved the same result with the help of the method of enlargement of obstacles. Formula (0.3) is also true if one replaces the soft obstacles W by hard obstacles, which immediately kill the Brownian particles if they hit such an obstacle (traps) (see Sznitman, 1998, Theorem 4.5.3). In the setting, of rarefied traps results have been obtained by Bolthausen (1990), Sznitman (1990), Bolthausen and den Hollander (1994), and van den Berg et al. (2001) (by scaling arguments the situation of rarefied traps can be viewed to be equivalent to that of shrinking hard obstacles). Here we study a slightly different problem, instead of rarefying hard obstacles we scale the soft obstacles.

There are two different physical motivations to consider scaled random potentials:

On the one hand, the ground state energy of a quantum mechanical particle in a random potential is given by the principal eigenvalue of the corresponding Schrödinger operator. For weak Poissonian potentials, one might guess the effect of the random potential to be simply a shift of the ground state energy by the average value of the potential. This guess looks reasonable especially if the typical de Broglie wavelengths of the quantum particle are large compared to the length scale of fluctuations in the potential. Strong random potentials might be able to confine the particle, leading to a total energy shift less than the average of the random potential. One goal of this article is to examine potentials of critical strength, where we expect these two scenarios to compete.

On the other hand, the principal eigenvalue of a Schrödinger operator with a non-negative potential may be interpreted as the long-time survival rate of a Brownian particle; now the potential is interpreted as a spatially dependent absorption rate. Here the question arises whether the long-time survival rate equals the averaged absorption rate: for weak absorption, one might guess that it does not pay off to try to avoid the obstacles; then Brownian particles just see an averaged potential. For strong potentials, one expects specific strategies that avoid strong absorption to be superior. Again the critical scale, where the picture for weak potentials just starts to break down, interests us most. The distinction between “weak” and “strong” potentials depends on the time horizon; this motivates us to make the scaling of the potential dependent on the time horizon.

More formally, for a scaling function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$, and $\beta>0$, we examine the asymptotic behavior of

$$\mathbb{E}\otimes E_0\left[\exp\left\{-\beta\varphi(t)^{-2}\int_0^t V(Z_s,\omega)\,\mathrm{d}s\right\}\right]\quad \text{as } t\to\infty. \tag{0.4}$$

We interpret the expected value (0.4) physically as the expected survival probability of a Brownian particle up to time t when moving in a random spatially dependent absorbing medium. The absorption rate is then given by $\beta\varphi(t)^{-2}V(\cdot,\omega)$.

Let us intuitively explain the effect of changing the time horizon: Consider a Brownian particle in a constant weakly absorbing random medium (say $\varphi=1$ and β being a very small constant). For short time-horizons, the absorbing random medium may be viewed as a small perturbation which hardly modifies the behavior of Brownian motion. The opposite is true for large time horizons: the Donsker–Varadhan (resp. Sznitman's) results tell us that non-absorbed particles get confined in "holes" (arising as large deviations) in the Poissonian cloud. We are interested in intermediate (critical) time scales, where on the one hand the surviving Brownian particles need to adapt their behavior considerably to the obstacles, but on the other hand the Donsker–Varadhan picture does not yet fully apply. Certainly, this intermediate time scale depends on the strength β of the absorbing medium: the critical time horizon for constant potentials increases as the absorption rate decreases.

The examination of (0.4) is related to the study of the annealed problem for the principal Dirichlet eigenvalue of the random Schrödinger operator $H_{\beta\varphi(t)^{-2}V}\stackrel{\text{def}}{=}-\frac{1}{2}\Delta+\beta\varphi(t)^{-2}V$: let us give a heuristic argument for the leading order in (0.4) being determined by a principal eigenvalue:

$$\begin{aligned}E_0\left[\exp\left\{-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s,\omega)\,\mathrm{d}s\right\}\right] &= \mathrm{e}^{-(tH_{\beta\varphi(t)^{-2}V})}1(0)\\ &\text{“}=\text{”}\sum_i \phi_{i,t}(0)\langle\phi_{i,t},1\rangle\mathrm{e}^{-t\lambda_{i,t}},\end{aligned} \tag{0.5}$$

where the $\lambda_{i,t}\geqslant 0$ are the "eigenvalues" of $H_{\beta\varphi(t)^{-2}V}$ and $\phi_{i,t}$ the corresponding "eigenfunctions". Heuristically, the sum in (0.5) is dominated by its ground state contribution ($i=0$) if the corresponding coefficient $\phi_{0,t}(0)\langle\phi_{0,t},1\rangle$ is asymptotically not too small.

So let us consider the bottom $\lambda_{\beta\varphi^{-2}(t)V}(U)$ of the spectrum of $H_{\beta\varphi(t)^{-2}V}$ over a non-empty open subset U of $\mathbb{R}^d$; more generally, for any measurable function $F:\mathbb{R}^d\to\mathbb{R}$ which is bounded from below the ground state energy with potential F is defined by

$$\lambda_F(U)\stackrel{\text{def}}{=}\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2+\int_U F\phi^2\,\mathrm{d}x:\ \phi\in C_c^\infty(U),\ \|\phi\|_2=1\right\}. \tag{0.6}$$

For $l:\mathbb{R}_+\to\mathbb{R}_+$ and $t>0$ we define $\mathscr{T}_{l(t)}\stackrel{\text{def}}{=}(-l(t),l(t))^d$. The logarithmic moment generating function of a Poissonian point process is defined as follows ($\beta>0$):

$$\Lambda_\phi(-\beta)\stackrel{\text{def}}{=}\log\mathbb{E}\left[\exp\left\{-\beta\int_{\mathbb{R}^d}\phi^2\,\mathrm{d}\omega\right\}\right]=\int_{\mathbb{R}^d}(\mathrm{e}^{-\beta\phi^2}-1)\,\mathrm{d}x \tag{0.7}$$

for $\phi \in \Phi \overset{\text{def}}{=} \{\phi \in H^{1,2}(\mathbb{R}^d)$: ϕ is compactly supported, continuous, and $\|\phi\|_2 = 1\}$. Then

$$J(\beta) \overset{\text{def}}{=} \inf\{\tfrac{1}{2}\|\nabla\phi\|_2^2 - \Lambda_\phi(-\beta)\colon \phi \in \Phi\}. \tag{0.8}$$

Finally, we define the constant $\tilde{c}(d,1)$: Let λ_d be the principal Dirichlet eigenvalue of $-\Delta/2$ on the d-dimensional unit ball $B_1(0)$. Then

$$\tilde{r}_d \overset{\text{def}}{=} \left(\frac{2\lambda_d}{d|B_1(0)|}\right)^{1/(d+2)}, \tag{0.9}$$

$$\tilde{c}(d,1) \overset{\text{def}}{=} \inf_{r>0}\left(\frac{\lambda_d}{r^2} + r^d|B_1(0)|\right) = \frac{\lambda_d}{\tilde{r}_d^2} + \tilde{r}_d^d|B_1(0)|. \tag{0.10}$$

These quantities have already been introduced by Sznitman (1998), formulas (4.5.30)–(4.5.32).

0.2. Results

Our first main result is the following theorem ("$a(t) \gg b(t)$" means that $a(t)/b(t) \to \infty$ as $t \to \infty$):

Theorem 0.1. *For $d \geqslant 1$, we choose $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ and $l: \mathbb{R}_+ \to \mathbb{R}_+$ such that $l(t) \gg (\varphi(t) \vee t^{1/(d+2)})$ and $\log l(t) \ll t(\varphi(t) \vee t^{1/(d+2)})^{-2}$. For $\beta > 0$ we have*

(a) *If $t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}$, then*

$$\lim_{t\to\infty} -\frac{\varphi(t)^2}{t}\log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] = \beta. \tag{0.11}$$

(b) *If $\varphi(t) = t^{1/(d+2)}$, then*

$$\lim_{t\to\infty} -t^{-d/(d+2)}\log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] = J(\beta). \tag{0.12}$$

(c) *If $\varphi(t) \ll t^{1/(d+2)}$, then*

$$\lim_{t\to\infty} -t^{-d/(d+2)}\log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] = \tilde{c}(d,1). \tag{0.13}$$

This result tells us that the critical scale, on which we may observe a phase transition, is $\varphi(t) = t^{1/(d+2)}$. It should be contrasted with the results obtained in the quenched case (see formulas (0.7), (0.15) and (0.16) in Merkl and Wüthrich, 2001a). As the special case $\varphi(t) = 1$, (0.13) contains the result (0.3) by Donsker–Varadhan.

For the annealed partition sum we obtain a similar behavior:

Theorem 0.2. *For $d \geqslant 1$ and $\beta > 0$ we have*

(a) *If $t^{1/(d+2)} \ll \varphi(t) \ll (t/\log t)^{1/2}$, then*

$$\lim_{t\to\infty} -\frac{\varphi(t)^2}{t}\log \mathbb{E}\otimes E_0\left[\exp\left(-\beta\varphi(t)^{-2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right] = \beta. \tag{0.14}$$

(b) *If* $\varphi(t)=t^{1/(d+2)}$, *then*

$$\lim_{t\to\infty} -t^{-d/(d+2)}\log \mathbb{E}\otimes E_0\left[\exp\left(-\beta\varphi(t)^{-2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]=J(\beta). \tag{0.15}$$

(c) *If* $\varphi(t)\ll t^{1/(d+2)}$, *then*

$$\lim_{t\to\infty} -t^{-d/(d+2)}\log \mathbb{E}\otimes E_0\left[\exp\left(-\beta\varphi(t)^{-2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]=\tilde{c}(d,1). \tag{0.16}$$

The upper bound in (0.16) is already contained in the proof of the upper bound in (0.3).

Our next results prove that on the scale $\varphi(t)=t^{1/(d+2)}$ we have a phase transition in dimensions $d\geqslant 2$ but not in dimension $d=1$.

Theorem 0.3. *For* $d\geqslant 2$, *there is a critical point* $\beta_c(d)>0$ *such that*

$$J(\beta)=\beta \quad \textit{for } 0<\beta\leqslant\beta_c(d), \tag{0.17}$$

$$J(\beta)<\beta \quad \textit{for } \beta>\beta_c(d). \tag{0.18}$$

However, this phase transition picture does not hold in dimension $d=1$, as the following theorem shows:

Theorem 0.4. *Assume* $d=1$. *Then for all* $\beta>0$, $J(\beta)<\beta$. *There are positive constants* $\tilde{C}_1\leqslant C_1$ *and* b_1 *such that for all* $\beta\in(0,b_1)$:

$$\beta-C_1\beta^4\leqslant J(\beta)\leqslant\beta-\tilde{C}_1\beta^4. \tag{0.19}$$

As a consequence, $J(\beta)$ in dimension $d=1$ is not proportional to β for small values of β: formally we may write $\beta_c(1)=0$. One should compare Theorems 0.3 and 0.4 with Theorems 0.3 and 0.4 of Merkl and Wüthrich (2001a). The remarkable thing is that in the annealed case we observe the critical dimension $d=2$ for having a phase transition, while the critical dimension in the quenched case equals $d=4$.

The next theorem plays an analogous role for the annealed problem as Theorem 0.2 in Merkl and Wüthrich (2001a) does in the quenched context:

Theorem 0.5. *For any dimension* $d\geqslant 1$ *there are positive constants* $C_2(d)$, $C_3(d)$, *and* $b_2(d)$ *such that for all* $\beta\geqslant b_2$ *the following bounds hold*:

$$\tilde{c}(d,1)-C_2\sqrt{\frac{\log\beta}{\beta}}\leqslant J(\beta)\leqslant\tilde{c}(d,1)-\frac{C_3}{\sqrt{\beta}}. \tag{0.20}$$

This theorem shows that in the limit $\beta\to\infty$ one asymptotically approaches the Donsker–Varadhan picture for unscaled potentials; one may compare this with (0.16) and (0.3).

0.3. Interpretation and heuristics

Let us compare our result (0.15) and Theorem 0.3 with the situation of rarefied hard obstacles: van den Berg et al. (2001), Section 1.6, Corollary 1 states for $d \geqslant 3$, $a > 0$, $c > 0$ and some constants $\kappa_a > 0$, $c_a^* > 0$:

$$\lim_{t\to\infty} -t^{-(d-2)/d} \log E_0[\exp\{-ct^{-2/d}|W^a(t)|\}] \begin{cases} = \kappa_a c & \text{for } 0 < c \leqslant c_a^*, \\ < \kappa_a c & \text{for } c > c_a^*. \end{cases} \tag{0.21}$$

Here $W^a(t) = \bigcup_{0\leqslant s\leqslant t} B_a(Z_s)$ denotes the Wiener sausage with radius a up to time t. In order to translate this into a statement about Brownian motion among rarefied traps, we introduce the hard obstacle potential $V_{\mathrm{h}}(x,\omega) = \int W_{\mathrm{h}}(x-y)\omega(\mathrm{d}y)$, where $W_{\mathrm{h}}(x) = \infty$ for $|x| \leqslant a$, $W_{\mathrm{h}}(x) = 0$ else. Furthermore, $\mathbb{E}_v$ denotes the expectation operator for a Poissonian point process ω over $\mathbb{R}^d$ with constant intensity v. Using these notations and the convention $\exp\{-\infty\} = 0$, we rewrite the left-hand side of (0.21) in the following form:

$$\lim_{t\to\infty} -t^{-(d-2)/d} \log \mathbb{E}_{ct^{-2/d}} \otimes E_0 \left[\exp\left\{-\int_0^t V_{\mathrm{h}}(Z_s,\omega)\,\mathrm{d}s\right\}\right]. \tag{0.22}$$

Comparing (0.21/0.22) with (0.15), we see that the two models show a similar critical behavior for certain power law scalings, although the critical scaling exponents do not coincide.

Let us intuitively compare the annealed picture with the quenched counterpart, which was examined in Merkl and Wüthrich (2001a,b): In the quenched picture, we look for large deviations in a given typical Poissonian cloud. The volume scale of such fluctuations in a box of size t is roughly $\log t$, e.g. this is typically the volume scale of the largest ball receiving no Poissonian point at all. On the other hand, in the annealed picture, untypical Poissonian clouds may (and will frequently) dominate the scenario: If the gain for the survival of the Brownian particle in untypically large deviations of the Poissonian cloud exceeds the probability costs for producing such a deviation, then such untypically large deviations in the Poissonian cloud will contribute much more to the annealed partition sum than typical configurations. This makes it intuitively plausible why the length scale $t^{1/(d+2)}$ in the annealed picture is much larger than the corresponding scale $(\log t)^{1/d}$ of the quenched picture: Although fluctuations on this power-law length scale $t^{1/(d+2)}$ are untypical and thus are irrelevant for the quenched picture, they are still frequent enough to yield the leading term in the annealed partition sum.

Roughly speaking, asymptotics (0.3) corresponds to the following strategy: Consider Poissonian configurations ω for which a large ball $B_r(0)$ remains obstacle-free, and confine the Brownian particle not to leave this ball up to a large time t. The cost to empty the ball (i.e. the negative logarithm of the corresponding probability) is roughly $|B_r(0)| = r^d|B_1(0)|$, up to small boundary effects. Furthermore, the cost for the confinement of the Brownian particle is roughly $t\lambda_0(B_r(0)) = tr^{-2}\lambda_0(B_1(0))$. Minimizing the total costs $r^d|B_1(0)| + tr^{-2}\lambda_0(B_1(0))$, one ends up with a optimal radius with the scaling $r = \tilde{r}_d t^{1/(d+2)}$, and the optimized total costs just yield the leading order in (0.3).

We reconsider the strategy to remove obstacles from a ball and to confine the Brownian particle to it: Making the obstacles weak enough, one intuitively expects that this strategy cannot remain optimal. In order to develop some intuition for the mechanisms behind the three cases in Theorem 0.2, let us do a rough, even a little oversimplified heuristic comparison: we only partially empty a ball. Suppose the ball $B_r(0)$ receives only about a fraction μ of the expected number of Poissonian points. The costs to produce such a large deviation are roughly $|B_r(0)|(\mu\log\mu - \mu + 1)$. In this simplified heuristic consideration, we replace the Poissonian potentials inside the ball by a smeared out constant potential $\mu\beta\varphi^{-2}(t)$, and we still confine the Brownian particles to stay inside the ball $B_r(0)$ up to the time horizon t. We end up with the total costs

$$r^d|B_1(0)|(\mu\log\mu - \mu + 1) + tr^{-2}\lambda_0(B_1(0)) + t\mu\beta\varphi^{-2}(t). \tag{0.23}$$

As a consequence of the diverging derivative of $\mu\log\mu-\mu+1$ for $\mu\to 0$ we obtain: For fixed t, the optimal total costs cannot be reached for $\mu=0$; thus it is favorable not to fully empty a ball. This need not be true in the limit as $t\to\infty$: three different regimes may occur here:

- First, the optimum in the total costs can be reached in the limit as $t\to\infty$ with $\mu\to 1$ and $rt^{-1/(d+2)}\to\infty$; then the strategy to "localize" Brownian motion in a ball-shaped large deviation in the Poissonian cloud does not pay off; Brownian motion in a constant averaged potential will do at least as good. In this regime, only the third summand in the total costs (0.23) remains asymptotically. This regime occurs if $t\beta\varphi^{-2}(t)$ is small enough.
- Second, the optimum as $t\to\infty$ in the total costs may be obtained in the limit $\mu\to 0$ when r is scaled proportional to $t^{1/(d+2)}$. In this regime one simply recovers the Donsker–Varadhan picture; here the first two summands in the total costs (0.23) are of comparable order, while the third one is negligible. This regime occurs if $t\beta\varphi^{-2}(t)$ is large enough.
- Third, the optimum can consist of a critical competition between all three summands in the total costs, none of them being negligible. This can occur only if we choose r proportional to $t^{1/(d+2)}$ and $\beta\varphi(t)^{-2}$ proportional to $t^{-2/(d+2)}$, with the proportionality constant β being large enough; then all summands in these total costs have the same scaling behavior as $t\to\infty$. In our simplified picture, this corresponds to a partially emptied ball and Brownian motion being localized in it. In a more realistic picture, the ball is replaced by a spherically symmetric optimal density pattern for Poissonian obstacles, and the strict confinement of Brownian particles is replaced by the introduction of an "inward drift", which softly drives the Brownian particle toward the center of the density pattern. In Section 2, we analyze the "costs" for the introduction of an "inward drift" for the Brownian particle, which is equivalent to a change of the underlying measure.

Some words of caution may be in place: When talking about a "optimal strategy", this refers only to the leading order of the costs. If the leading order of a strategy is the optimal one, this does not imply that such a strategy is seen in a "typical" picture, conditioned on survival of the Brownian particle. Other contributions with the same leading order of the costs might do equally well or even better with lower costs in higher order terms. In this article, we do not rigorously identify the typical behavior of Brownian particles among scaled obstacles conditioned on survival.

Let us explain how this article is organized: Formulas (0.11) and (0.12) are proved in Section 1, Lemmas 1.1 and 1.2. Claims (0.13) and (0.16) are also proved in Section 1. In Section 2 we show (0.14) and (0.15). The proof of Theorems 0.3 and 0.4 is prepared in Section 3.1, but it is completed at the end of Section 3, whereas the proof of Theorem 0.5 is given in Section 3.2 (Lemmas 3.3 and 3.4).

A remark concerning our sign conventions: *upper bounds* for the ground state energy of a Schrödinger operator and for free energies correspond to *lower bounds* for survival probabilities of Brownian motion among obstacles and vice versa. Our terminology for "upper" and "lower" bounds is motivated by the physical interpretation in terms of "energies". Readers who are used to think in the language of large deviations should keep this inversion of signs in mind.

1. Asymptotic behavior of the principal Dirichlet eigenvalue

In this section we prove Theorem 0.1 and the lower bound in (0.16). For $r > 0$, $y \in \mathbb{R}^d$, and a function ϕ the scaling operator S_y^r is defined by

$$(S_y^r \phi)(x) \stackrel{\text{def}}{=} r^{-d/2} \phi((x-y)/r). \tag{1.1}$$

Lemma 1.1. *For all positive scaling functions $l(t) \gg \varphi(t)$ and $\beta > 0$:*

$$\limsup_{t\to\infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] \leqslant \beta. \tag{1.2}$$

For the special scaling function $\varphi(t) = t^{1/(d+2)}$ the following bound holds:

$$\limsup_{t\to\infty} - t^{-d/(d+2)} \log \mathbb{E}[\exp(-t\lambda_{\beta t^{-2/(d+2)}V}(\mathscr{T}_{l(t)}))] \leqslant J(\beta). \tag{1.3}$$

Remark. For the special scaling function $\varphi(t) = t^{1/(d+2)}$, inequality (1.3) is stronger than the general inequality (1.2) at least in some cases; see Theorem 0.3. Inequality (1.2) can be proven using Jensen's inequality. We do not proceed in this way, but treat instead (1.2) and (1.3) at the same time below.

Proof of Lemma 1.1. Choose $\phi \in \Phi$. Since ϕ is compactly supported and $l(t) \gg \varphi(t)$ we have for all sufficiently large t that the function $S_0^{\varphi(t)}\phi$ is supported in $\mathscr{T}_{l(t)}$. We estimate for these large $t > 0$, using the notation $*$ for the convolution operator, and $W_r^-(x) \stackrel{\text{def}}{=} r^d W(-rx)$:

$$\begin{aligned}
&\log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] \\
&\quad \geqslant \log \mathbb{E}\left[\exp\left(-\frac{t}{2}\|\nabla(S_0^{\varphi(t)}\phi)\|_2^2 - \frac{\beta t}{\varphi(t)^2}\int_{\mathscr{T}_{l(t)}} (S_0^{\varphi(t)}\phi)^2 V \,\mathrm{d}x\right)\right] \\
&\quad = -\frac{t}{2\varphi(t)^2}\|\nabla\phi\|_2^2 + \log \mathbb{E}\left[\exp\left(-\frac{\beta t}{\varphi(t)^2}\int_{\mathbb{R}^d} W_1^- * (S_0^{\varphi(t)}\phi)^2 \,\mathrm{d}\omega\right)\right]
\end{aligned}$$

$$= -\frac{t}{2\varphi(t)^2}\|\nabla\phi\|_2^2 + \int_{\mathbb{R}^d}\left(\exp\left(-\frac{\beta t}{\varphi(t)^2}W_1^- * (S_0^{\varphi(t)}\phi)^2\right) - 1\right)\mathrm{d}x$$

$$= -\frac{t}{2\varphi(t)^2}\|\nabla\phi\|_2^2 + \varphi(t)^d\int_{\mathbb{R}^d}\left(\exp\left(-\frac{\beta t}{\varphi(t)^{d+2}}W_{\varphi(t)}^- * \phi^2\right) - 1\right)\mathrm{d}x$$

$$\geqslant -\frac{t}{2\varphi(t)^2}\|\nabla\phi\|_2^2 - \frac{\beta t}{\varphi(t)^2}\int_{\mathbb{R}^d} W_{\varphi(t)}^- * \phi^2\,\mathrm{d}x$$

$$= -\frac{t}{\varphi(t)^2}\left(\frac{1}{2}\|\nabla\phi\|_2^2 + \beta\right), \tag{1.4}$$

we used $\|\phi\|_2^2 = 1$, $W \geqslant 0$, $\|W_r^-\|_1 = \|W\|_1 = 1$ in the last step. Estimate (1.4) implies

$$\limsup_{t\to\infty} -\frac{\varphi(t)^2}{t}\log\mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] \leqslant \frac{1}{2}\|\nabla\phi\|_2^2 + \beta. \tag{1.5}$$

The gradient term $\|\nabla\phi\|_2^2$ can be made arbitrarily small; hence (1.5) implies claim (1.2). To derive (1.3) in the case $\varphi(t) = t^{1/(d+2)}$, we proceed as follows: Using that ϕ is continuous, and $W \geqslant 0$, $\|W_r^-\|_1 = \|W\|_1 = 1$, we get $W_{\varphi(t)}^- * \phi^2 \overset{t\to\infty}{\to} \phi^2$ pointwise. Using the dominated convergence theorem one sees

$$\int_{\mathbb{R}^d}(\exp(-\beta W_{\varphi(t)}^- * \phi^2) - 1)\,\mathrm{d}x \overset{t\to\infty}{\to} \Lambda_\phi(-\beta) \tag{1.6}$$

and hence, using the fifth line in (1.4):

$$\limsup_{t\to\infty} -t^{-d/(d+2)}\log\mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] \leqslant \frac{1}{2}\|\nabla\phi\|_2^2 - \Lambda_\phi(-\beta). \tag{1.7}$$

Using definition (0.8) of J, claim (1.3) of Lemma 1.1 follows from (1.7). □

Lemma 1.2. *Assume that the scaling function φ satisfies either $t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}$ or $\varphi(t) = t^{1/(d+2)}$. Further assume that the scaling function l fulfills $\log l(t) \ll t/\varphi(t)^2$. Then for all $\beta > 0$:*

$$\liminf_{t\to\infty} -\frac{\varphi(t)^2}{t}\log\mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))]$$

$$\geqslant \begin{cases} J(\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \beta & \text{for } t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}. \end{cases} \tag{1.8}$$

Proof. We use some notations from Merkl and Wüthrich (2001a, Section 2.1): for $M > 0$, $V^M \overset{\text{def}}{=} V \wedge M$, and for $\zeta > 0$, $j \in \zeta\mathbb{Z}^d$ we set $K_j(\zeta) \overset{\text{def}}{=} j + [0,\zeta)^d$ and $\omega^\zeta \overset{\text{def}}{=} \sum_{j\in\zeta\mathbb{Z}^d} 1_{\{\omega(K_j)\geqslant 1\}}\delta_j$; here δ_j means the Dirac measure located at j. We set $\tilde{V}^\zeta(x,\omega) \overset{\text{def}}{=} \int_{\mathbb{R}^d} W(x-y)\omega^\zeta(\mathrm{d}y)$; this is the Bernoulli version of the Poissonian potential $V(\cdot,\omega)$. Finally for $\psi \in H^{1,2}(\mathbb{R}^d)$ and a measurable function F we abbreviate $\mathscr{E}_F(\psi) \overset{\text{def}}{=} \|\nabla\psi\|_2^2/2 + \int F\psi^2\,\mathrm{d}x$ (whenever the right-hand side is well defined). The following notation deviates a little from the one chosen in Merkl and Wüthrich (2001a), since we have now $l(t)$ instead of t as the length scale of the universe box: $Y_{R,t}^{\varphi,l} \overset{\text{def}}{=} \{y \in d^{-1/2}R\varphi(t)\mathbb{Z}^d\colon B_{R\varphi(t)}(y) \cap \mathscr{T}_{l(t)} \neq \emptyset\}$. Lemmas 2.3 and 2.4 in Merkl and Wüthrich (2001a), especially estimate (2.40) there, show: For positive β, η and ζ there are $M > 0$, $R \geqslant 1$, a finite set $\Psi \subseteq \{\psi \in C_c^1(\overline{B_{R+1}}(0)) \colon \|\psi\|_2 = 1\}$ and $t_0 > 0$ such that for all $t > t_0$ and $\omega \in \Omega$:

$$\lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)}) \geqslant \min_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathcal{E}_{\beta\varphi(t)^{-2}V^M}(S_y^{\varphi(t)}\psi) - 3\varphi(t)^{-2}\eta \tag{1.9}$$

and

$$\varphi(t)^2 \max_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} (\mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi) - \mathcal{E}_{\beta\varphi(t)^{-2}V^M}(S_y^{\varphi(t)}\psi))$$

$$\leqslant 2\beta \max_{\psi \in \Psi}(\|\psi\|_\infty + \|\nabla\psi\|_\infty)^2 \sqrt{d}\zeta^{1-d} R^d |B_4(0)| \varphi(t)^{-1} \leqslant \eta. \tag{1.10}$$

We emphasize the following fact: t_0 does not depend on $\omega \in \Omega$, since the first estimate in (1.10), which coincides with the last estimate in (2.40) in Merkl and Wüthrich (2001a), is uniform in the Poissonian configuration ω. Eqs. (1.9) and (1.10) yield

$$\lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)}) \geqslant \min_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi) - 4\varphi(t)^{-2}\eta. \tag{1.11}$$

Therefore (again for $t \geqslant t_0$):

$$\begin{aligned}
&\log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)}))] \\
&\quad\leqslant \log \mathbb{E}\left[\max_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \exp(-t\mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi))\right] + 4t\varphi(t)^{-2}\eta \\
&\quad\leqslant \log \sum_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathbb{E}[\exp(-t\mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi))] + 4t\varphi(t)^{-2}\eta \\
&\quad\leqslant \sup_{\substack{y \in \mathbb{R}^d \\ \psi \in \Psi}} \log \mathbb{E}[\exp(-t\mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi))] \\
&\qquad + \log|Y_{R,t}^{\varphi,l}| + \log|\Psi| + 4t\varphi(t)^{-2}\eta.
\end{aligned} \tag{1.12}$$

We get for all $\psi \in \Psi$ and $y \in \mathbb{R}^d$ (cf. with (2.46) in Merkl and Wüthrich, 2001a):

$$\begin{aligned}
&\frac{\varphi(t)^2}{t} \log \mathbb{E}[\exp\{-t\mathcal{E}_{\beta\varphi(t)^{-2}\tilde{V}^\zeta}(S_y^{\varphi(t)}\psi)\}] \\
&\quad = -\frac{\varphi(t)^2}{2}\|\nabla S_y^{\varphi(t)}\psi\|_2^2 + \frac{\varphi(t)^2}{t} \log \mathbb{E}\left[\exp\left\{-\frac{\beta t}{\varphi(t)^2}\int_{\mathbb{R}^d}(S_y^{\varphi(t)}\psi)^2 \tilde{V}^\zeta \,\mathrm{d}x\right\}\right].
\end{aligned} \tag{1.13}$$

To estimate the expectation in the last expression, we proceed analogously to the quenched case, see Lemma 2.5 in Merkl and Wüthrich (2001a): We define the discretized version $v^\zeta \stackrel{\text{def}}{=} \zeta^d \sum_{j \in \zeta\mathbb{Z}^d} \delta_j$ of the Lebesgue measure, abbreviate $m \stackrel{\text{def}}{=} \zeta^{-d}(1-\mathrm{e}^{-\zeta^d}) \stackrel{\zeta\to 0}{\to} 1$, and use bound (2.45) in Merkl and Wüthrich (2001a), which is the following estimate for the Laplace transform of a Bernoulli process (discretized Poissonian point process): $\log \mathbb{E}[\exp(\int f \,\mathrm{d}\omega^\zeta)] \leqslant m\int(\mathrm{e}^f - 1)\,\mathrm{d}v^\zeta$.

The first summand on the right-hand side of (1.13) equals $-\|\nabla\psi\|_2^2/2$, while the second summand equals

$$\begin{aligned}
&\frac{\varphi(t)^2}{t}\log\mathbb{E}\left[\exp\left(-\frac{\beta t}{\varphi(t)^2}\int_{\mathbb{R}^d}(S_y^{\varphi(t)}\psi)^2 * W_1^- \,\mathrm{d}\omega^\zeta\right)\right]\\
&\quad\leqslant \frac{m\varphi(t)^2}{t}\int_{\mathbb{R}^d}\left(\exp\left\{-\frac{\beta t}{\varphi(t)^2}(S_y^{\varphi(t)}\psi)^2 * W_1^-\right\}-1\right)\mathrm{d}v^\zeta\\
&\quad= \frac{m\varphi(t)^{d+2}}{t}\int_{\mathbb{R}^d}\left(\exp\left\{\frac{-\beta t}{\varphi(t)^{d+2}}(S_{y/\varphi(t)}^1\psi)^2 * W_{\varphi(t)}^-\right\}-1\right)\mathrm{d}v^{\zeta/\varphi(t)}. \qquad (1.14)
\end{aligned}$$

We insert this bound in (1.13) and take the supremum over $y\in\mathbb{R}^d$ and the limit as $t\to\infty$; in this limit the above Riemann sum $\int\ldots\mathrm{d}v^{\zeta/\varphi(t)}$ converges to an integral. We get

$$\begin{aligned}
&\limsup_{t\to\infty}\sup_{y\in\mathbb{R}^d}\frac{\varphi(t)^2}{t}\log\mathbb{E}[\exp\{-t\mathscr{E}_{\beta\varphi(t)^{-2}\tilde V^\zeta}(S_y^{\varphi(t)}\psi)\}]\\
&\quad\leqslant\begin{cases}-\frac{1}{2}\|\nabla\psi\|_2^2+m\Lambda_\psi(-\beta) & \text{for } \varphi(t)=t^{1/(d+2)},\\ -\frac{1}{2}\|\nabla\psi\|_2^2-m\beta & \text{for } \varphi(t)\gg t^{1/(d+2)}.\end{cases} \qquad (1.15)
\end{aligned}$$

The assumptions $\log l(t)\ll t/\varphi(t)^2$ and $\varphi(t)\ll t^{1/2}$ imply $\log|Y_{R,t}^{\varphi,l}|+\log|\Psi|\ll t/\varphi(t)^2$. Combining (1.12) and (1.15) we obtain

$$\begin{aligned}
&\liminf_{t\to\infty}\;-\frac{\varphi(t)^2}{t}\log\mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))]\\
&\quad\geqslant\begin{cases}\min_{\psi\in\Psi}\frac{1}{2}\|\nabla\psi\|_2^2-m\Lambda_\psi(-\beta)-4\eta & \text{for } \varphi(t)=t^{1/(d+2)},\\ \min_{\psi\in\Psi}\frac{1}{2}\|\nabla\psi\|_2^2+m\beta-4\eta & \text{for } t^{1/2}\gg\varphi(t)\gg t^{1/(d+2)},\end{cases}\\
&\quad\geqslant\begin{cases}mJ(\beta)-4\eta & \text{for } \varphi(t)=t^{1/(d+2)},\\ m\beta-4\eta & \text{for } t^{1/2}\gg\varphi(t)\gg t^{1/(d+2)}.\end{cases} \qquad (1.16)
\end{aligned}$$

Claim (1.8) of Lemma 1.2 now follows by taking the limits $\eta\to 0$ and $\zeta\to 0$, i.e. $m\uparrow 1$. Lemma 1.2 is proved. □

Proof of (0.13) and (0.16). For $\varphi(t)\ll t^{1/(d+2)}$ we have for all $\beta,\beta'>0$ and all large t:

$$\frac{\beta}{\varphi(t)^2}V\geqslant\frac{\beta'}{t^{2/(d+2)}}V. \qquad (1.17)$$

By monotonicity, this implies $\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})\geqslant\lambda_{\beta' t^{-2/(d+2)}V}(\mathscr{T}_{l(t)})$. Hence, using (0.12) and Theorem 0.5 (which is proven in Section 3, below):

$$\liminf_{t\to\infty}\;-t^{-d/(d+2)}\log\mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))]\geqslant J(\beta')\overset{\beta'\to\infty}{\to}\tilde c(d,1). \qquad (1.18)$$

An analogous monotonicity estimate also holds true for the partition sums; this proves the lower bounds in (0.13) and (0.16).

To prove the upper bounds, we set $r(t) \stackrel{\text{def}}{=} t^{1/(d+2)}\tilde{r}_d$ and choose a length scale $l(t) \gg r(t)$. Then we have (where a denotes the minimal radius such that the support of W is contained in the ball $\bar{B}_a(0)$):

$$\begin{aligned}
&\limsup_{t\to\infty} - t^{-d/(d+2)} \log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)}))] \\
&\quad\leqslant \limsup_{t\to\infty} - t^{-d/(d+2)} \log \mathbb{E}[\exp(-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})), \omega(B_{r(t)+a}(0)) = 0] \\
&\quad\leqslant \limsup_{t\to\infty} - t^{-d/(d+2)} \log \mathbb{E}[\exp(-t\lambda_{V\equiv 0}(B_{r(t)}(0))), \omega(B_{r(t)+a}(0)) = 0] \\
&\quad= \limsup_{t\to\infty} t^{-d/(d+2)}(t\lambda_d r(t)^{-2} - \log \mathbb{P}[\omega(B_{r(t)+a}(0)) = 0]) \\
&\quad= \lambda_d \tilde{r}_d^{-2} + \tilde{r}_d^d |B_1(0)| = \tilde{c}(d,1).
\end{aligned} \tag{1.19}$$

The proof of the upper bound (0.16) is the same as in Sznitman (1998), Theorem 4.5.3, (4.5.33)–(4.5.36). This finishes the proofs. □

2. Asymptotic behavior of the partition sum

In this section, we prove (0.14) and (0.15) of Theorem 0.2. The main tool to obtain upper bounds in (0.14) and (0.15) is a change of measure, which transforms Brownian motion into a (stationary) diffusion process: Using this diffusion process as "strategy" for the Brownian particle turns out to be optimal (at least in the leading order) for survival among scaled Poissonian obstacles.

Proof of the upper bounds in (0.14) and (0.15). We treat both cases at the same time. Alternatively, the proof of (0.14) could be treated separately, simply using Jensen's inequality.

Let $\phi \in \Phi$, $\phi \geqslant 0$. We first introduce a modification ϕ^ε of ϕ which is positive everywhere with an exponential decay at infinity: Let $\delta_1 \in C^\infty(\mathbb{R}^d)$ denote a fixed positive function with exponential decay at infinity and with $\|\delta_1\|_1 = 1$, to be explicit, say $\delta_1(x) = c_4 \mathrm{e}^{-|x|}$ with a positive constant c_4 for all x outside a compact subset of $\mathbb{R}^d$. For every multiindex n, we get the following bound on the nth derivative: There is a constant $c_5(n) > 0$ such that $|D_n\delta_1| \leqslant c_5(n)\delta_1$. For $\varepsilon > 0$, we define the following approximation to Dirac's δ: $\delta_\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon^{-d}\delta_1(x/\varepsilon)$. Let $\hat{f}(k) = \int \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$ denote the Fourier transform. Using the dominated convergence theorem we see

$$\|\phi * \delta_\varepsilon\|_2^2 = (2\pi)^{-d}\|\hat{\phi}(k)\hat{\delta}_1(\varepsilon k)\|_2^2 \stackrel{\varepsilon\to 0}{\to} (2\pi)^{-d}\|\hat{\phi}\|_2^2 = \|\phi\|_2^2 = 1 \tag{2.1}$$

and similarly

$$\|\nabla\phi * \delta_\varepsilon\|_2^2 \stackrel{\varepsilon\to 0}{\to} \|\nabla\phi\|_2^2. \tag{2.2}$$

Consequently $\phi^\varepsilon \stackrel{\text{def}}{=} \|\phi * \delta_\varepsilon\|_2^{-1}(\phi * \delta_\varepsilon)$ satisfies $\|\nabla\phi^\varepsilon\|_2^2 \stackrel{\varepsilon\to 0}{\to} \|\nabla\phi\|_2^2$, $\phi^\varepsilon > 0$, and there exist $\varepsilon_0 > 0$, $r_0 > 0$ and $c_6 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $x \in \mathbb{R}^d$ with $|x| > r_0$:

$\phi^{\varepsilon}(x) \leqslant c_6 e^{-|x|}$. We get for all $\beta > 0$, using the dominated convergence theorem once more:

$$\frac{1}{2}\|\nabla\phi^{\varepsilon}\|_2^2 - \int_{\mathbb{R}^d} (e^{-\beta(\phi^{\varepsilon})^2} - 1)\,dx \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\|\nabla\phi\|_2^2 - \int_{\mathbb{R}^d} (e^{-\beta\phi^2} - 1)\,dx. \tag{2.3}$$

For $\varepsilon > 0,\ t > 0$, we set $\phi_{\varepsilon,t} \overset{\text{def}}{=} S_0^{\varphi(t)}\phi^{\varepsilon}$. With ε, t being fixed for the moment, we define $b \overset{\text{def}}{=} \nabla \log \phi_{\varepsilon,t}$. By a change of measure, we introduce a diffusion process with drift $b(Z_s)$ over the finite time horizon $t < \infty$: the bounds on the derivatives of δ_1 imply

$$\sup_{x\in\mathbb{R}^d} |D_n b(x)| < \infty \tag{2.4}$$

for every multiindex n; especially the Novikov condition (see e.g. Karatzas and Shreve, 1991, Corollary 3.5.13)

$$E_x\left[\exp\left(\frac{1}{2}\int_0^t |b(Z_s)|^2\,ds\right)\right] < \infty \tag{2.5}$$

is satisfied. By the Cameron–Martin–Girsanov theorem,

$$\tilde{Z}_s = Z_s - \int_0^s b(Z_u)\,du \tag{2.6}$$

is a d-dimensional Brownian motion with respect to the probability measure

$$Q_x = \exp\left\{\int_0^t b(Z_s)\,dZ_s - \frac{1}{2}\int_0^t |b(Z_s)|^2\,ds\right\} P_x. \tag{2.7}$$

We denote the expectation operator with respect to Q_x by E_x^Q, while the symbol E_x is reserved for expectations with respect to P_x. We claim that $\phi_{\varepsilon,t}^2\,dx$ is an invariant distribution with respect to the transformed diffusion process, i.e. for every non-negative measurable test function $f: \mathbb{R}^d \to \mathbb{R}$ we have for all $s \in [0,t]$:

$$\int_{\mathbb{R}^d} \phi_{\varepsilon,t}(x)^2 E_x^Q[f(Z_s)]\,dx = \int_{\mathbb{R}^d} \phi_{\varepsilon,t}(x)^2 f(x)\,dx. \tag{2.8}$$

It suffices to prove (2.8) for $f \in C_c^{\infty}(\mathbb{R}^d)$: In this case, bounds (2.4) on the derivative of the drift imply that

$$g(x,s) \overset{\text{def}}{=} E_x^Q[f(Z_s)] \tag{2.9}$$

is a classical solution of the Cauchy problem

$$\frac{\partial g}{\partial s} = \tfrac{1}{2}\Delta g + b\cdot\nabla g, \tag{2.10}$$

$$g(x,0) = f(x) \tag{2.11}$$

with bounded derivatives in x and s of every order (see e.g. Freidlin, 1985, Section 5.3, Theorems 3.1 and 3.2; Friedman, 1976, Sections 6.4, 6.5). We use the heat equation

(2.10) and integrate partially to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^d} g(x,s)\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x &= \int_{\mathbb{R}^d}\frac{\partial g}{\partial s}(x,s)\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\left(\frac{1}{2}\Delta g(x,s)+\frac{\nabla\phi_{\varepsilon,t}(x)}{\phi_{\varepsilon,t}(x)}\cdot\nabla g(x,s)\right)\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\nabla g(x,s)\phi_{\varepsilon,t}(x)(\nabla\phi_{\varepsilon,t}(x)-\nabla\phi_{\varepsilon,t}(x))\,\mathrm{d}x=0.\end{aligned} \tag{2.12}$$

The boundary terms of the partial integration vanish, since $\phi_{\varepsilon,t}$ and its derivatives decay exponentially at infinity, while g and its derivatives are bounded. Our claim (2.8) is a consequence of (2.12).

The measure P_x is absolutely continuous with respect to Q_x with the Radon–Nikodym derivative

$$\begin{aligned}\frac{\mathrm{d}P_x}{\mathrm{d}Q_x} &= \exp\left\{-\int_0^t b(Z_s)\,\mathrm{d}Z_s+\frac{1}{2}\int_0^t|b(Z_s)|^2\,\mathrm{d}s\right\}\\ &= \exp\left\{-\int_0^t b(Z_s)\,\mathrm{d}\tilde{Z}_s-\frac{1}{2}\int_0^t|b(Z_s)|^2\,\mathrm{d}s\right\}.\end{aligned} \tag{2.13}$$

We remark that the stochastic integral in (2.13) remains unchanged when the underlying probability measure P_x is replaced by the equivalent measure Q_x. By translational invariance of the Poisson process we get

$$\begin{aligned}&\mathbb{E}\otimes E_0\left[\exp\left(-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]\\ &\quad=\mathbb{E}\left[\int_{\mathbb{R}^d}E_x\left[\exp\left(-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x\right].\end{aligned} \tag{2.14}$$

Define $Q\overset{\text{def}}{=}\int_{\mathbb{R}^d}Q_x[\cdot]\,\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x$ to be the probability measure which makes $(Z_s)_{0\leqslant s\leqslant t}$ a (stationary) diffusion process with starting distribution $\phi_{\varepsilon,t}^2$ and drift b. We use (2.13), Jensen's inequality, and the fact that $(\int_0^s b(Z_u)\,\mathrm{d}\tilde{Z}_u)_{0\leqslant s\leqslant t}$ is a Q-martingale in the following estimate:

$$\begin{aligned}&\int_{\mathbb{R}^d}E_x\left[\exp\left(-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x\\ &\quad=E^Q\left[\exp\left(-\int_0^t b(Z_s)\,\mathrm{d}\tilde{Z}_s-\frac{1}{2}\int_0^t|b(Z_s)|^2\,\mathrm{d}s-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right)\right]\\ &\quad\geqslant\exp\left\{E^Q\left[-\int_0^t b(Z_s)\,\mathrm{d}\tilde{Z}_s-\frac{1}{2}\int_0^t|b(Z_s)|^2\,\mathrm{d}s-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right]\right\}\\ &\quad=\exp\left\{-\int_0^t E^Q\left[\frac{1}{2}|b(Z_s)|^2+\frac{\beta}{\varphi(t)^2}V(Z_s)\right]\,\mathrm{d}s\right\}\\ &\quad\overset{(2.8)}{=}\exp\left\{-t\int_{\mathbb{R}^d}\left(\frac{1}{2}|b(x)|^2+\frac{\beta}{\varphi(t)^2}V(x)\right)\phi_{\varepsilon,t}(x)^2\,\mathrm{d}x\right\}\\ &\quad=\exp\left\{-\frac{t}{2}\,\|\nabla\phi_{\varepsilon,t}\|_2^2-\frac{\beta t}{\varphi(t)^2}\int_{\mathbb{R}^d}V\phi_{\varepsilon,t}^2\,\mathrm{d}x\right\}.\end{aligned} \tag{2.15}$$

Combining (2.15) with (2.14), we obtain, using the dominated convergence theorem (recall that ϕ^ε decays exponentially fast at infinity) and (2.1)–(2.3):

$$
\begin{aligned}
&-\frac{\varphi(t)^2}{t}\log \mathbb{E}\otimes E_0\left[\exp\left(-\int_0^t \frac{\beta}{\varphi(t)^2}V(Z_s)\,\mathrm{d}s\right)\right]\\
&\quad\leqslant \frac{\varphi(t)^2}{2}\|\nabla\phi_{\varepsilon,t}\|_2^2-\frac{\varphi(t)^2}{t}\int_{\mathbb{R}^d}\left(\exp\left\{-\frac{\beta t}{\varphi(t)^2}\phi_{\varepsilon,t}^2 * W_1^-\right\}-1\right)\mathrm{d}x\\
&\quad=\frac{1}{2}\|\nabla\phi^\varepsilon\|_2^2-\frac{\varphi(t)^{d+2}}{t}\int_{\mathbb{R}^d}\left(\exp\left\{-\frac{\beta t}{\varphi(t)^{d+2}}(\phi^\varepsilon)^2 * W_{\varphi(t)}^-\right\}-1\right)\mathrm{d}x\\
&\quad\overset{t\to\infty}{\longrightarrow}\begin{cases}\frac{1}{2}\|\nabla\phi^\varepsilon\|_2^2-\int_{\mathbb{R}^d}\left(\exp\{-\beta(\phi^\varepsilon)^2\}-1\right)\mathrm{d}x & \text{for } \varphi(t)=t^{1/(d+2)},\\ \frac{1}{2}\|\nabla\phi^\varepsilon\|_2^2+\beta\int_{\mathbb{R}^d}(\phi^\varepsilon)^2\,\mathrm{d}x & \text{for } \varphi(t)\gg t^{1/(d+2)}\end{cases}\\
&\quad\overset{\varepsilon\to 0}{\longrightarrow}\begin{cases}\frac{1}{2}\|\nabla\phi\|_2^2-\Lambda_\phi(-\beta) & \text{for } \varphi(t)=t^{1/(d+2)},\\ \frac{1}{2}\|\nabla\phi\|_2^2+\beta & \text{for } \varphi(t)\gg t^{1/(d+2)}.\end{cases}
\end{aligned}
\tag{2.16}
$$

When we optimize over $\phi\in\Phi$, $\phi\geqslant 0$, we get the two upper bounds in (0.14) and (0.15). Note that we may restrict the infimum in definition (0.8) of J to non-negative test functions ϕ. $\square$

Proof of the lower bounds in (0.14) and (0.15). We treat both cases at the same time. We choose any scaling function $l:\mathbb{R}_+\to\mathbb{R}_+$ with $\log l(t)\ll t/\varphi(t)^2$ and $l(t)\gg t/\varphi(t)$ as $t\to\infty$; one possible choice is $l(t)=t$.

Let $T_{l(t)}\overset{\text{def}}{=}\inf\{s: Z_s\notin\mathscr{T}_{l(t)}\}$ denote the exit time from the box $\mathscr{T}_{l(t)}$. Since the potential V is bounded on compact domains, the random Schrödinger operator $-\Delta/2+\beta\varphi(t)^{-2}V$ is essentially self-adjoint on $C_c^\infty(\mathscr{T}_{l(t)})$; for fixed $\beta>0$ and scaling functions φ and l we denote its closure by H_t. The self-adjoint operator H_t is bounded from below: $H_t\geqslant\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})\mathbf{1}$; hence $\mathrm{e}^{-tH_t}:L^2(\mathscr{T}_{l(t)})\to L^2(\mathscr{T}_{l(t)})$ is a bounded, self-adjoint operator with

$$\left\|\mathrm{e}^{-tH_t}\right\|_{L^2\to L^2}\leqslant \mathrm{e}^{-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})}, \tag{2.17}$$

we also refer to Sznitman (1998, Proposition 1.3.3). Let $f\in C_c^\infty(\mathbb{R}^d)$, $f\geqslant 0$, $\|f\|_1=1$ be any fixed test function. We choose a fixed $r>0$ such that f is supported in $\mathscr{T}_r$. We get for $l(t)>r$, using the Feynman–Kac representation of e^{-tH_t}:

$$
\begin{aligned}
&\int_{\mathbb{R}^d} f(x)E_x\left[\exp\left\{-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right\}\right]\mathrm{d}x\\
&\quad\leqslant\int_{\mathbb{R}^d} f(x)\left(P_x[T_{l(t)}\leqslant t]+E_x\left[\exp\left\{-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right\}, T_{l(t)}>t\right]\right)\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^d} f(x)P_x[T_{l(t)}\leqslant t]\,\mathrm{d}x+\left\langle 1_{\mathscr{T}_{l(t)}},\mathrm{e}^{-tH_t}f\right\rangle\\
&\quad\leqslant P_0[T_{l(t)-r}\leqslant t]+\|1_{\mathscr{T}_{l(t)}}\|_2\|\mathrm{e}^{-tH_t}\|_{L^2\to L^2}\|f\|_2\\
&\quad\leqslant 4d\exp\{-(l(t)-r)^2/(2t)\}+(2l(t))^{d/2}\|f\|_2\exp\left\{-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})\right\}.
\end{aligned}
\tag{2.18}
$$

Using Lemma 1.2 and $l(t)^2/t \gg t/\varphi(t)^2$, we see that the first summand in the last sum is negligible as $t \to \infty$ compared to the expected value of the second one. We get, using translation invariance of the Poisson process, Lemma 1.2, and $\log l(t) \ll t/\varphi(t)^2$:

$$\begin{aligned}
&\liminf_{t\to\infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \otimes E_0 \left[\exp\left\{-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right\}\right] \\
&\quad = \liminf_{t\to\infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E}\left[\int_{\mathbb{R}^d} f(x) E_x\left[\exp\left\{-\frac{\beta}{\varphi(t)^2}\int_0^t V(Z_s)\,\mathrm{d}s\right\}\right]\right] \mathrm{d}x \\
&\quad \geqslant \liminf_{t\to\infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E}\left[\exp\left\{-t\lambda_{\beta\varphi(t)^{-2}V}(\mathscr{T}_{l(t)})\right\}\right] \\
&\quad = \begin{cases} J(\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \beta & \text{for } t^{1/(d+2)} \ll \varphi(t) \ll (t/\log t)^{1/2}. \end{cases}
\end{aligned} \tag{2.19}$$

This finishes the proof of the lower bounds in (0.14) and (0.15). □

3. Analysis of the variational principle

In this section, we prove Theorems 0.3–0.5. We start with some easy facts:

Lemma 3.1. *J is a concave, monotonically increasing function with $J(\beta) \leqslant \beta$.*

Proof. For all $\phi \in \Phi$, the functions $\mathbb{R}^+ \ni \beta \mapsto \frac{1}{2}\|\nabla\phi\|_2^2 - \Lambda_\phi(-\beta)$ are concave and monotonically increasing. Consequently, the infimum over $\phi \in \Phi$ has these properties, too. Furthermore, the fact $-\Lambda_\phi(-\beta) \leqslant \beta$ implies $J(\beta) \leqslant \inf_{\phi\in\Phi} \frac{1}{2}\|\nabla\phi\|_2^2 + \beta = \beta$. □

For $\phi \in \Phi$, $r > 0$ we introduce the scaled version $\phi_r(x) \stackrel{\text{def}}{=} r^{-d/2}\phi(x/r)$; its basic properties are collected in (3.26)–(3.27) in Merkl and Wüthrich (2001b). We get for every $\beta > 0$ and every dimension $d \geqslant 1$:

$$\begin{aligned}
J(\beta) &= \inf\{\tfrac{1}{2}\|\nabla\phi_r\|_2^2 - \Lambda_{\phi_r}(-\beta)\colon r > 0,\ \phi \in \Phi,\ \|\nabla\phi\|_2 = 1\} \\
&= \inf\left\{\frac{1}{2r^2} - r^d \Lambda_\phi(-r^{-d}\beta)\colon r > 0,\ \phi\in\Phi,\ \|\nabla\phi\|_2 = 1\right\}.
\end{aligned} \tag{3.1}$$

Finally, we denote by r_d the radius of a d-dimensional ball of volume d. We set $c(d,1) \stackrel{\text{def}}{=} r_d^{-2}\lambda_d$; this is the principal Dirichlet eigenvalue of $-\Delta/2$ on such a ball $B_{r_d}(0)$. These $c(d,1)$ and r_d play an analogous role in the quenched case as $\tilde{c}(d,1)$ and $\tilde{r}_d$ in the annealed problem (see Merkl and Wüthrich, 2001a).

3.1. The phase transition picture

Lemma 3.2. (a) *For $d = 1$, there are positive constants $c_1 \leqslant C_1$ and c_7 such that for all $\beta > 0$:*

$$\beta - C_1\beta^4 \leqslant J(\beta) \leqslant \beta - c_1\beta^4 + c_7\beta^7. \tag{3.2}$$

(b) *For $d \geqslant 2$, there is $b_3(d) > 0$ such that for all $\beta \in (0, b_3(d)]$ we have $J(\beta) \geqslant \beta$.*

Remark. We write the upper bound in (3.2) in this form, since it naturally arises in the proof.

Proof of Lemma 3.2. Let $\phi \in \Phi$ with $\|\nabla\phi\|_2 = 1$, and $r > 0$. For $y = \beta\phi_r(x)^2 \geqslant 0$ we integrate the inequalities $y - y^2/2 \leqslant 1 - e^{-y} \leqslant y - y^2/2 + y^3/6$ over $x \in \mathbb{R}^d$ and use $\|\nabla\phi_r\|_2^2 = r^{-2}$, $\|\phi_r\|_2^2 = 1$, $\|\phi_r\|_4^4 = r^{-d}\|\phi\|_4^4$, $\|\phi_r\|_6^6 = r^{-2d}\|\phi\|_6^6$ to get for $\beta > 0$:

$$\frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^d}\|\phi\|_4^4 \leqslant \frac{1}{2}\|\nabla\phi_r\|_2^2 - \Lambda_{\phi_r}(-\beta) \leqslant \frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^d}\|\phi\|_4^4 + \frac{\beta^3}{6r^{2d}}\|\phi\|_6^6. \tag{3.3}$$

The upper bound in (3.2) in the case $d = 1$ follows from the upper bound in (3.3) and from (3.1) when we set $r = 2\|\phi\|_4^{-4}\beta^{-2}$, $c_1 = \|\phi\|_4^8/8$, and $c_7 = \|\phi\|_4^8\|\phi\|_6^6/24$ for any fixed ϕ. To derive the lower bounds in the dimensions $d \leqslant 3$, we use bound (3.8) in Merkl and Wüthrich (2001a), which tells us (for $d < 4$)

$$c_8(d) \stackrel{\text{def}}{=} \sup\{\|\phi\|_4^4 : \phi \in \Phi, \|\nabla\phi\|_2 = 1\} < \infty. \tag{3.4}$$

We insert this bound into the lower bound in (3.3) and minimize over $r > 0$ (in view of (3.1)):

- For $d = 1$ we get the lower bound in (3.2) with $C_1 = c_8(d)^2/8$ and the minimizing radius $r = 2c_8(1)^{-1}\beta^{-2}$.
- For $d = 2$ and $\beta \leqslant b_3(2) \stackrel{\text{def}}{=} c_8(2)^{-1/2}$ we get $J(\beta) \geqslant \beta$; here the infimum over r of the lower bound in (3.3) is reached in the limit $r \to \infty$.
- For $d = 3$ we distinguish 2 cases:
  Case 1: If $r \leqslant (2\beta)^{-1/2}$, then we get, using $\Lambda_\phi(-r^{-d}\beta) \leqslant 0$:

$$\frac{1}{2r^2} - r^d\Lambda_\phi(-r^{-d}\beta) \geqslant \frac{1}{2r^2} \geqslant \beta. \tag{3.5}$$

  Case 2: We suppose $r > (2\beta)^{-1/2}$. For $\beta \leqslant b_3(3) \stackrel{\text{def}}{=} 2^{-1/5}c_8(3)^{-2/5}$ we get

$$\frac{1}{2r^2} - r^d\Lambda_\phi(-r^{-d}\beta) \geqslant \frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^3}c_8(3) \geqslant \beta + \frac{1}{2r^2}(1 - 2^{1/2}c_8(3)\beta^{5/2}) \geqslant \beta. \tag{3.6}$$

For $d \geqslant 4$, the case $r \leqslant (2\beta)^{-1/2}$ is treated the same way as in the case $d = 3$. In the case $r > (2\beta)^{-1/2}$ we proceed as follows: Eq. (3.19) in Merkl and Wüthrich (2001a) tells us for $\sigma \leqslant 0$ (recall $d \geqslant 4$): $\Lambda_\phi(\sigma) \leqslant \sigma + c_9|\sigma|^{d/(d-2)}$, where $c_9(d) \stackrel{\text{def}}{=} (\frac{2}{d})^{(2d-2)/(d-2)}$ $(d-2)^{2/(2-d)}\pi^{(d+1)/(2-d)}\Gamma(d/2 + 1/2)^{2/(d-2)}$, hence

$$r^d\Lambda_\phi(-r^{-d}\beta) \leqslant -\beta + c_9 r^{2d/(2-d)}\beta^{d/(d-2)}. \tag{3.7}$$

We define $b_3(d) \stackrel{\text{def}}{=} 2^{-d/(d+2)}c_9^{(2-d)/(d+2)}$. We use bound (3.7), the hypothesis $\beta \leqslant b_3$, and the assumption $r > (2\beta)^{-1/2}$ to get

$$\frac{1}{2r^2} - r^d\Lambda_\phi(-r^{-d}\beta) \geqslant \beta + r^{-2}\left(\frac{1}{2} - c_9 r^{4/(2-d)}\beta^{d/(d-2)}\right)$$

$$\begin{aligned}
&\geqslant \beta + r^{-2}\left(\frac{1}{2} - c_9(2\beta)^{2/(d-2)}\beta^{d/(d-2)}\right) \\
&= \beta + \frac{1}{2r^2}\left(1 - \left(\frac{\beta}{b_3}\right)^{(d+2)/(d-2)}\right) \\
&\geqslant \beta. \qquad (3.8)
\end{aligned}$$

The claim of Lemma 3.2 now follows for all dimensions using (3.1). □

3.2. Asymptotics in the large-β-region

Lemma 3.3. *There are positive constants $C_3(d)$ and $b_4(d)$ such that for all $\beta \geqslant b_4$ the following upper bound holds*:

$$J(\beta) \leqslant \tilde{c}(d,1) - \frac{C_3}{\sqrt{\beta}}. \qquad (3.9)$$

Proof. By the upper bound in Lemma B.1 in Appendix B of Merkl and Wüthrich (2001a) there are positive constants b_5 and c_{10} such that for all $\beta_1 \geqslant b_5$ there is a test function $\psi \in \Phi$ that fulfills

$$\frac{1}{2}\|\nabla\psi\|_2^2 + \beta_1\|\psi 1_{\mathbb{R}^d\setminus B_{r_d}(0)}\|_2^2 \leqslant c(d,1) - \frac{c_{10}}{\sqrt{\beta_1}}. \qquad (3.10)$$

We define $C_3 \overset{\text{def}}{=} (r_d/\tilde{r}_d)^3 c_{10}$, $b_4 \overset{\text{def}}{=} (r_d/\tilde{r}_d)^2 b_5$. Given $\beta \geqslant b_4$, we set $\beta_1 = (\tilde{r}_d/r_d)^2\beta \geqslant b_5$, choose $\psi \in \Phi$ as in (3.10), and scale: $\phi(x) = (r_d/\tilde{r}_d)^{d/2}\psi(xr_d/\tilde{r}_d)$. We obtain

$$\frac{1}{2}\|\nabla\phi\|_2^2 + \beta\|\phi 1_{\mathbb{R}^d\setminus B_{\tilde{r}_d}(0)}\|_2^2 \leqslant c(d,1)\left(\frac{\tilde{r}_d}{r_d}\right)^{-2} - \frac{C_3}{\sqrt{\beta}}. \qquad (3.11)$$

Using the inequality $-(\mathrm{e}^{-\xi} - 1) \leqslant \xi \wedge 1$ we get, using (0.9), (0.10), and the definition of r_d and of $c(d,1)$ in the last step:

$$\begin{aligned}
J(\beta) &\leqslant \frac{1}{2}\|\nabla\phi\|_2^2 - \int_{\mathbb{R}^d}(\mathrm{e}^{-\beta\phi^2} - 1)\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\|\nabla\phi\|_2^2 + \beta\|\phi 1_{\mathbb{R}^d\setminus B_{\tilde{r}_d}(0)}\|_2^2 + \|1_{B_{\tilde{r}_d}(0)}\|_2^2 \\
&\leqslant c(d,1)\left(\frac{\tilde{r}_d}{r_d}\right)^{-2} - \frac{C_3}{\sqrt{\beta}} + \left(\frac{\tilde{r}_d}{r_d}\right)^{d} \mathrm{d} = \tilde{c}(d,1) - \frac{C_3}{\sqrt{\beta}}. \qquad (3.12)
\end{aligned}$$

Lemma 3.3 is proved. □

Lemma 3.4. *There are positive constants $C_2(d)$ and $b_6(d)$ such that for all $\beta \geqslant b_6$ the following lower bound holds*:

$$J(\beta) \geqslant \tilde{c}(d,1) - C_2\sqrt{\frac{\log\beta}{\beta}}. \qquad (3.13)$$

Proof. This time we use the lower bound in Lemma B.1 in Appendix B of Merkl and Wüthrich (2001a). By the same scaling argument as the one leading to (3.11) we obtain: There are positive constants $c_{11}(d)$ and $b_7(d)$ such that for every radius $s > 0$, every $\beta_1 \geqslant s^{-2} b_7$, and every test function $\phi \in \Phi$ we have

$$\tfrac{1}{2}\|\nabla\phi\|_2^2 + \beta_1 \|\phi 1_{\mathbb{R}^d \setminus B_s(0)}\|_2^2 \geqslant c(d,1) r_d^2 s^{-2} - c_{11}\beta_1^{-1/2} s^{-3}. \tag{3.14}$$

For fixed β_1 and ϕ, the left-hand side in (3.14) is a monotonically decreasing function of s. We choose a constant radius $\rho_d > 0$ so small that

$$c(d,1) r_d^2 \rho_d^{-2} > \tilde{c}(d,1), \tag{3.15}$$

and then a constant $b_8(d) \geqslant \rho_d^{-2} b_7$ so large that

$$c(d,1) r_d^2 \rho_d^{-2} - c_{11} b_8^{-1/2} \rho_d^{-3} \geqslant \tilde{c}(d,1). \tag{3.16}$$

Hence we get for all $\beta_1 \geqslant b_8$ and all $r \geqslant 0$, using (3.14) and (3.16):

$$\frac{1}{2}\|\nabla\phi\|_2^2 + \beta_1 \|\phi 1_{\mathbb{R}^d \setminus B_r(0)}\|_2^2 \geqslant \begin{cases} \tilde{c}(d,1) & \text{for } r < \rho_d, \\ c(d,1) r_d^2 r^{-2} - c_{11}\beta_1^{-1/2} r^{-3} & \text{for } r \geqslant \rho_d. \end{cases} \tag{3.17}$$

We choose a constant $b_9 \geqslant 4$ so large that $b_9/\log b_9 \geqslant b_8$. Let $\beta \geqslant b_9$ and $\phi \in \Phi$. As in the quenched case (Lemma 3.6 in Merkl and Wüthrich (2001a)) we use a rearrangement inequality: Let $\phi^\circ \in \Phi$ denote the radially symmetric non-increasing rearrangement of $\phi \in \Phi$ (see Lieb and Loss, 1997, Section 3.3). Then $\Lambda_\phi = \Lambda_{\phi^\circ}$, and $\|\nabla\phi^\circ\|_2 \leqslant \|\nabla\phi\|_2$; see Lemma 7.17 in Lieb and Loss (1997). Let $r \geqslant 0$ denote the maximal radius such that $\beta\phi^\circ(x)^2 > \frac{1}{2}\log\beta$ holds for all $x \in B_r(0)$; consequently $\beta\phi^\circ(x)^2 \leqslant \frac{1}{2}\log\beta$ holds for all $x \in \mathbb{R}^d \setminus B_r(0)$, since ϕ° is radially symmetric non-increasing. We use the inequality

$$1 - e^y \geqslant \begin{cases} \dfrac{\beta^{-1/2} - 1}{\frac{1}{2}\log\beta} y & \text{for } -\frac{1}{2}\log\beta \leqslant y \leqslant 0, \\ 1 - \beta^{-1/2} & \text{for } y \leqslant -\frac{1}{2}\log\beta \end{cases} \tag{3.18}$$

and we abbreviate $\beta_1 = (1 - \beta^{-1/2})\beta/(\frac{1}{2}\log\beta) \geqslant \beta/\log\beta \geqslant b_8$ (recall $\beta \geqslant b_9 \geqslant 4$) in the following estimate:

$$\begin{aligned} &\frac{1}{2}\|\nabla\phi\|_2^2 - \Lambda_\phi(-\beta) \\ &\quad \geqslant \frac{1}{2}\|\nabla\phi^\circ\|_2^2 + \int_{\mathbb{R}^d} (1 - e^{-\beta(\phi^\circ)^2})\,dx \\ &\quad \geqslant \frac{1}{2}\|\nabla\phi^\circ\|_2^2 + \beta_1 \|\phi^\circ 1_{\mathbb{R}^d \setminus B_r(0)}\|_2^2 + (1 - \beta^{-1/2})|B_r(0)| \\ &\quad \overset{(3.17)}{\geqslant} \begin{cases} \tilde{c}(d,1) & \text{for } r < \rho_d, \\ c(d,1) r_d^2 r^{-2} - c_{11}\beta_1^{-1/2} r^{-3} + (1 - \beta^{-1/2})|B_1(0)| r^d & \text{for } r \geqslant \rho_d. \end{cases} \end{aligned} \tag{3.19}$$

We estimate the last expression for sufficiently large β in the case $r \geqslant \rho_d$: We abbreviate $c_{12}(d) \overset{\text{def}}{=} c_{11}\rho_d^{-1} c(d,1)^{-1} r_d^{-2}$, $c_{13}(d) \overset{\text{def}}{=} c_{12} \vee 1$, and $C_2(d) \overset{\text{def}}{=} \tilde{c}(d,1) c_{13}$. Then we choose $b_6 \geqslant b_9$ so large that $c_{13}(\log b_6)^{1/2} b_6^{-1/2} \leqslant 1$. Assume $\beta \geqslant b_6$. We estimate (recall the definition (0.10) of $\tilde{c}(d,1)$):

$$\begin{aligned} & c(d,1)r_d^2 r^{-2} - c_{11}\beta_1^{-1/2} r^{-3} + (1-\beta^{-1/2})|B_1(0)|r^d \\ & \geqslant (1 - c_{12}\beta_1^{-1/2})c(d,1)r_d^2 r^{-2} + (1-\beta^{-1/2})|B_1(0)|r^d \\ & \geqslant (1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2})(c(d,1)r_d^2 r^{-2} + |B_1(0)|r^d) \\ & \geqslant (1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2})\tilde{c}(d,1) = \tilde{c}(d,1) - C_2(\log\beta)^{1/2}\beta^{-1/2}. \end{aligned} \tag{3.20}$$

We used in the last inequality $1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2} \geqslant 0$, which follows from the choice of b_6 and from $(\log\beta)\beta^{-1} \leqslant (\log b_6)b_6^{-1}$; recall $\beta \geqslant b_6 \geqslant 4$. Estimates (3.19), (3.20) and definition (0.8) of J together yield the claim (3.13) of Lemma 3.4. □

Proof of Theorem 0.3. On the one hand, Lemmas 3.2 and 3.1 imply $J(\beta) = \beta$ for $0 < \beta \leqslant b_3$, $d \geqslant 2$. On the other hand, Lemma 3.3 has the consequence $J(\beta) < \beta$ for large β. These two facts together with the concavity of J (see Lemma 3.1) imply Theorem 0.3. □

Proof of Theorem 0.4. It only remains to show $J(\beta) < \beta$ for all $\beta > 0$ in dimension $d = 1$. To see this, we observe $J(\beta) \xrightarrow{\beta\downarrow 0} 0$ and $J(\beta) < \beta$ for sufficiently small $\beta > 0$ as a consequence of the bounds (3.2), and use the concavity of J. □

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References

Bolthausen, E., 1990. On the volume of the Wiener sausage. Ann. Probab. 18, 1576–1582.

Bolthausen, E., den Hollander, F., 1994. Survival asymptotics for Brownian motion in a Poisson field of decaying traps. Ann. Probab. 27, 160–176.

Donsker, M., Varadhan, S.R.S., 1975. Asymptotics for the Wiener sausage. Comm. Pure Appl. Math. 28, 525–565.

Freidlin, M., 1985. Functional Integration and Partial Differential Equations, Annals of Mathematical Studies, Vol. 109. Princeton University Press, Princeton.

Friedman, A., 1976. Stochastic Differential Equations and Applications, Vol. 1. Academic Press, New York.

Karatzas, I., Shreve, S.E., 1991. Brownian Motion and Stochastic Calculus, 2nd Edition, Springer GTM, Vol. 113. Springer, New York.

Lieb, E.H., Loss, M., 1997. Analysis. Graduate Studies in Mathematics, Vol. 14. American Mathematical Society, Providence, RI.

Merkl, F., Wüthrich, M.V., 2001a. Infinite volume asymptotics of the ground state energy in a scaled Poissonian potential. Ann. Inst. H. Poincaré, to appear.

Merkl, F., Wüthrich, M.V., 2001b. Phase transition of the principal Dirichlet eigenvalue in a scaled Poissonian potential. Probab. Theory Related Fields 119 (4), 475–507.

Sznitman, A.S., 1990. Long time asymptotics for the shrinking Wiener sausage. Comm. Pure Appl. Math. 43, 809–820.

Sznitman, A.S., 1998. Brownian Motion, Obstacles and Random Media, Springer Monographs in Mathematics. Springer, Berlin, Heidelberg.

van den Berg, M., Bolthausen, E., den Hollander, F., 2001. Moderate deviations for the volume of the Wiener sausage. Ann. Math. 153, 355–406.